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FREQUENCY AND PHASE CONTROL OF THE RESONANCE OSCILLATIONS OF A NON-LINEAR SYSTEM UNDER CONDITIONS OF UNCERTAINTY†

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A norm-bounded locally optimal control is constructed which minimizes the frequency and phase deviations from resonance in a non-linear system affected by bounded perturbations. It is shown that this control is independent of the form of the perturbation and the structure of the conservative part of the system. As an example, frequency and phase controls of forced oscillations in a system of two weakly coupled oscillators are constructed.

It is well known that the analysis of the perturbed motion in the neighbourhood of resonance is similar to the analysis of the motion of "an equivalent pendulum" [1, 2]. The phase and frequency of the pendulum oscillations correspond, respectively, to the phase and frequency deviations from the resonance surface of the initial system (phase and frequency detuning). The phase plane of the pendulum is divided into the domains of oscillations and rotation separated by the separatrix, which is interpreted as the separatrix of resonance. The passage through the separatrix from the domain of bounded oscillations to the domain of rotation corresponds to unlimited frequency detuning and breakdown of resonance. The purpose of the control is to prevent the system from escaping from the admissible domain as a result of the perturbation.

This model enables one to use the well-developed asymptotic methods of controlling oscillatory system [3, 4]. Formally, the averaging procedure can be applied to a fairly wide range of systems. However, in practice the optimal control problem is prohibitively difficult if a non-linear system is considered over a relatively long time interval until it escapes from the near-resonance region. In practice, this problem can be resolved explicitly only for a single-degree-of-freedom system [5, 6].

The solution can be simplified if the motion is considered within a bounded domain over a relatively short time interval. A similar problem for the stochastically induced escape from the near-resonance region was discussed in [7].

This paper substitutes the locally optimal control problem [8] for the problem of a control preventing the escape from the near-resonance. Locally optimal control minimizes the phase and frequency detuning at each instant of time. The use of locally optimal control does not require a detailed specification of the properties of the perturbation.

1. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Consider a two-frequency system with a scalar slow variable. An extension to a multidimensional system is discussed in Section 2.

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The equations of motion are reduced to the standard form with slow and fast variables

$$\begin{aligned} \dot{x} &= \varepsilon f(x, \theta_1, \theta_2) + \varepsilon^n F(x, \theta_1, \theta_2)u + \varepsilon\Delta(x, \theta_1, \theta_2, \xi(t)), \quad x \in X, \ u \in U \\ \dot{\theta}_i &= \omega_i(x) + \varepsilon f_i(x, \theta_1, \theta_2) + \varepsilon^n G_i(x, \theta_1, \theta_2)u + \varepsilon\Delta_i(x, \theta_1, \theta_2, \xi(t)) \end{aligned} \tag{1.1}$$

where $\theta_i(\mathrm{mod}\,2\pi)$, $i = 1, 2$, X is an open domain, U is a compactum in R_1 and $\varepsilon > 0$ is a small parameter. The exponent n in the coefficient ε^n is chosen in such a way that control remains weak but counteracting the external perturbation in accordance with the requirements of the problem.

The right-hand sides of system (1.1) are assumed to be 2π-periodic in the fast phases θ_1 and θ_2 and smooth enough in all the variables. The smoothness assumptions imply that a solution of system (1.1) exists and the requisite transformations are valid for any admissible control. The perturbation $\xi(t)$ is assumed to be bounded, that is $|\xi(t)| \le \xi_0$ for all $t \ge t_0$. If $\xi(t)$ is a random process, it is assumed to be bounded with probability 1.

We will specify the resonance relations between the system frequencies [1, 2]. Consider the time average of the function $f(x, \theta_1(t), \theta_2(t))$ in the form

$$\Phi(x, \omega_1, \omega_2) = \lim_{T \to \infty} \frac{1}{T}\int_0^T f(x, \omega_1 t + \alpha_1, \omega_2 t + \alpha_2)dt$$

The function $\Phi(x, \omega_1, \omega_2)$ is assumed to be independent of the arbitrary phases α_1, α_2 and discontinuous in the line

$$\rho(x) = m_1\omega_1(x) + m_2\omega_2(x) = 0 \tag{1.2}$$

where m_1 and m_2 are certain integers not equal to zero simultaneously. Equation (1.2) determines the resonance relation between the system frequencies. We also assume that Eq. (1.2) has a unique solution x^* such that

$$\rho(x^*) = 0, \quad d\rho(x^*)/dx = r \neq 0 \tag{1.3}$$

Suppose the time averages of the functions $\Delta(x, \theta_1(t), \theta_2(t), \xi(t))$ and $G_i(x, \theta_1(t), \theta_2(t))$ do not yield new resonance relations in a small neighbourhood of the point x^*, that is they are continuous for any relations between the frequencies $\omega_1(x^*)$ and $\omega_2(x^*)$.

Suppose the unperturbed system exhibits stable resonance oscillations with frequencies $\omega_1(x^*)$ and $\omega_2(x^*)$ satisfying Eq. (1.2). The purpose of the control is to keep the frequencies in the near-resonance region when there are perturbations resulting in the variable x deviating from the reference value $x = x^*$ and violating the resonance relation (1.2). We will formulate this requirement as a control problem. We will specify an admissible domain of motion and construct a control which minimizes the deviations of the frequency from the resonance values.

Following the standard approach [1, 2], we introduce the variables υ and φ, representing the frequency and phase detuning, respectively. We write

$$\mu\upsilon = \rho(x) = m_1\omega_1(x) + m_2\omega_2(x), \quad \mu = \varepsilon^{1/2}, \quad \varphi = m_1\theta_1 + m_2\theta_2 \tag{1.4}$$

Conditions (1.3) and (1.4) imply the following relations in the near-resonance region

$$x = X(\mu\upsilon) = x^* + \mu x_1 + \mu^2, \quad x_1 = r^{-1}\upsilon; \quad \theta_1 = \theta, \quad \theta_2 = m_2^{-1}(\varphi - m_1\theta) \tag{1.5}$$

Substituting Eqs (1.4) and (1.5) into system (1.1) we obtain equations in standard form with the small parameter μ

$$\begin{aligned} \dot{\upsilon} &= \mu[f^*(\varphi, \theta) + \Delta^*(\varphi, \theta, \xi(t))] + \mu^{2n-1}F^*(\varphi, \theta)u + \mu^2 R_1(\upsilon, \varphi, \theta, u, \xi(t), \mu) \\ \dot{\varphi} &= \mu\upsilon + \mu^{2n}G^*(\varphi, \theta)u + \mu^2 R_2(\upsilon, \varphi, \theta, u, \xi(t), \mu) \\ \dot{\theta} &= \omega^* + \mu\Omega\upsilon + \mu^{2n}G_1^*(\varphi, \theta)u + \mu^2 R_3(\upsilon, \varphi, \theta, u, \xi(t), \mu) \end{aligned} \tag{1.6}$$

where $\theta = \theta_1$, $\omega^* = \omega_1(x^*)$ and $\Omega = \omega_{1x}(x^*)$. The coefficients on the right-hand side of system (1.6) are defined by the relations

$$Z^*(\varphi, \theta) = r^{-1}Z(x^*, \theta, \theta_2(\varphi, \theta))$$
$$G^*(\varphi, \theta) = m_1G_1(x^*, \theta, \theta_2(\varphi, \theta)) + m_2G_2(x^*, \theta, \theta_2(\varphi, \theta)) \tag{1.7}$$

where $Z = (f, \Delta, F, G_1)$ and $Z^* = (f^*, \Delta^*, F^*, G_1^*)$. The residual terms R_i on the right-hand side of (1.6) vanish as $\mu \to 0$, and their explicit form is unimportant.

Let us define the admissible domain of motion. We will consider the conservative subsystem of system (1.6), namely,

$$\dot{\upsilon} = \mu\beta(\varphi), \quad \dot{\varphi} = \mu\upsilon \tag{1.8}$$

where $\beta(\varphi) = \langle f^*(\varphi, \theta)\rangle$, and the symbol $\langle\ \rangle$ denotes averaging in the fast variable θ over the period. Equations (1.8) describe the motion of the conservative system with the Hamiltonian

$$H^{\mu}(\varphi, \upsilon) = \mu H(\varphi, \upsilon), \quad H(\varphi, \upsilon) = U(\varphi) + \upsilon^2/2 \tag{1.9}$$

where $U(\varphi)$ is a periodic potential function such that $U_{\varphi}(\varphi) = -\beta(\varphi)$. The solutions φ^* and φ^s of the equation $\beta(\varphi) = 0$ determine the minimum and maximum of the function $U(\varphi)$, respectively. We have

$$U(\varphi^*) = 0, \quad U_{\varphi\varphi}(\varphi^*) = -\beta_{\varphi}(\varphi^*) = -k^2 < 0$$

at the minimum point, and

$$U(\varphi^s) = H^s, \quad U_{\varphi\varphi}(\varphi^s) = -\beta_{\varphi}(\varphi^s) > 0$$

at the maximum point.

In the phase plane of the pendulum the domain of oscillations corresponds to the closed domain Σ confined to the homoclinic separatrix. The points $\varphi = \varphi^s$, $\upsilon = 0$ correspond to the vertices of the separatrix, and the steady-state point $\varphi = \varphi^*$, $\upsilon = 0$ determines the parameters of stable resonance in the unperturbed system. Passage through the separatrix from the domain of oscillations to the domain of rotation is associated with unlimited phase detuning and with breakdown of resonance. Hence the admissible domain of motion is defined by the conditions $(\upsilon, \varphi) \in \Sigma$.

This implies that control of the oscillation frequency in the near-resonance region can be interpreted as control of the perturbed motion of the pendulum within the admissible domain Σ. In this case, the energy of the pendulum (1.9) can be considered as a measure of the deviation of orbits (1.6) from the steady-state point.

Suppose $(\upsilon(t_0), \varphi(t_0)) \in \Sigma$ at the initial instant t_0. We construct a locally optimal control minimizing the derivative

$$J(u) = \dot{H} \tag{1.10}$$

for each t under the constraint $|u| \le U_0$. The locally optimal control is defined as [8]

$$u_{\text{opt}} = \underset{|u| \le U_0}{\operatorname{argmin}} J(u) \tag{1.11}$$

It follows from definitions (1.10) and (1.11) that control (1.11) provides the maximum decrement of the pendulum energy at each instant t. This corresponds to minimization of the pendulum deviations from the unperturbed state $\varphi = \varphi^*$, $\upsilon = 0$.

Calculating the derivative (1.10) by virtue of Eq. (1.6) we obtain

$$\begin{aligned}\dot{H} &= -\beta(\varphi)\dot{\varphi} + \upsilon\dot{\upsilon} = -\beta(\varphi)[\mu\upsilon + \mu^{2n}G^*(\varphi, \theta)u + \mu^2R_2] + \\ &+ \upsilon\{\mu[f^*(\varphi, \theta) + \Delta^*(\varphi, \theta, \xi(t))] + \mu^{2n-1}F^*(\varphi, \theta)u + \mu^2R_1\} = \\ &= \mu\upsilon[b(\varphi, \theta) + \Delta^*(\varphi, \theta, \xi(t)) + \mu^{2n-2}F^*(\varphi, \theta)u] - \mu^{2n}\beta(\varphi)G^*(\varphi, \theta)u + \mu^2R\end{aligned} \tag{1.12}$$

where $b(\varphi, \theta) = f^*(\varphi, \theta) - \beta(\varphi)$, $\langle b(\varphi, \theta)\rangle = 0$. The coefficient R comprises the residual terms, which have no effect on further transformations.

The introduction of the small parameter μ enables us to construct a near-optimal control u^* of relatively simple structure such that $u^* \to u_{opt}$ as $\mu \to 0$. We will consider the near-optimal control for two types of controlled systems.

1. $F(x, \theta_1, \theta_2) \neq 0$. In this case we let $n = 1$. Then, as $\mu \to 0$, the control term in the first of Eqs (1.6) is of the leading order, whereas the control terms in other equations are small. In view of the order relations among the terms on the right-hand side of Eq. (1.12), we obtain as $\mu \to 0$

$$u^* = -U_0 \operatorname{sign} F^*(\varphi, \theta) \operatorname{sign} \upsilon \tag{1.13}$$

Control (1.13) produces a moment which counteracts the frequency detuning. Formulae (1.7) and (1.13) yield a feedback control in the form

$$u^0(x, y, \theta_1, \theta_2) = -U_0 \operatorname{sign}[r^{-1}F(y, \theta_1, \theta_2)] \operatorname{sign} \upsilon \tag{1.14}$$

Substituting the control laws (1.13) or (1.14) into Eq. (1.1) and repeating all the above transformations, we obtain that Eqs (1.6) and the values of the functionals (1.10) are identical when $u = u^*$ and $u = u^0$. Controls (1.13) and (1.14) are independent of the perturbation and of the structure of the uncontrolled subsystem. The only parameter to be determined is the sign of r. Near-optimality of controls (1.13) and (1.14) can be proved in a standard way.

We will estimate the limiting control performance. Substituting controls (1.13) or (1.14) into system (1.16) and averaging all terms except the perturbation, we obtain a partially averaged system of the form

$$\begin{aligned} \dot{\upsilon}_0 &= \mu[\beta(\varphi_0) + \Delta^*(\varphi_0, \theta, \xi(t)) - U_0 f_0(\varphi_0) \operatorname{sign} \upsilon_0] \\ \dot{\varphi}_0 &= \mu \upsilon_0, \quad \dot{\theta} = \omega^* + \mu\Omega\upsilon_0 \end{aligned} \tag{1.15}$$

where $f_0(\varphi) = \langle |F^*(\varphi, \theta)|\rangle$, that is $f_0(\varphi) \geq 0$. The change in total energy of system (1.15) can be written as

$$\dot{E} = \mu[-\beta(\varphi_0)\dot{\varphi}_0 + \upsilon_0\dot{\upsilon}_0] = \mu[-U_0 f_0(\varphi_0)|\upsilon_0| + \Delta^*(\varphi_0, \theta, \xi(t))\upsilon_0] \tag{1.16}$$

It is obvious that the perturbation increases the deviations from the reference position but the control force decreases the deviations compared with the uncontrolled system.

In general, estimation of the limiting control performance is quite complicated. However, if $|\Delta^*(\varphi_0, \theta, \xi(t))| \leq \Delta_0$ and $U_0 f_0(\varphi_0) > \Delta_0$ for all admissible values of the variables φ_0, θ and $\xi(t)$, then $E(t) < 0$ at each instant t. This implies that the detuning decreases at each instant t, and $\varphi \to \varphi^*$ and $\upsilon \to 0$ as $t \to \infty$.

The estimate of the limiting performance can be improved if the perturbation properties are specified. Let $\xi(t)$ be a periodic or quasi-periodic process such that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Delta^*(\varphi, \omega^* t, \xi(t))dt = 0 \tag{1.17}$$

exists uniformly in $\varphi \in \Sigma$. Then, replacing the partially averaged system by the averaged one and calculating the total energy of the averaged system, we obtain

$$\dot{E} = -\mu U_0 f_0(\varphi_0)|\upsilon_0| < 0 \tag{1.18}$$

It is easy to see that the detuning decreases at each instant t, and $\varphi \to \varphi^*$ and $\upsilon \to 0$ as $t \to \infty$ for any $U_0 > 0$.

The physical interpretation of this result follows from Eq. (1.15). It means that the effect of controls (1.13) and (1.14) on the "equivalent pendulum" is similar to the effect of the maximum admissible Coulomb friction.

Under the assumptions made, the control directly affects the frequency detuning υ, whereas the phase detuning φ changes by virtue of Eqs (1.6) or (1.8). We define the function

$$H^{\upsilon} = \upsilon^2/2 \tag{1.19}$$

as a measure of the frequency detuning. We construct a control u, υ which minimizes the derivative $\dot{H}^{\upsilon}(t)$ at each instant t under the constraint $|u^{\upsilon}| \le U_0$, that is

$$J^{\upsilon}(u) = \dot{H}^{\upsilon}, \quad u^{\upsilon} = \underset{|u| \le U_0}{\operatorname{argmin}} J^{\upsilon}(u) \tag{1.20}$$

In the same way as before we find that

$$u^{\upsilon} = u^* = -U_0 \operatorname{sign} F^*(\varphi, \theta) \operatorname{sign} \upsilon \tag{1.21}$$

The control (1.21) and thus the controls (1.13) and (1.14) can be interpreted as frequency controls.

2. $F(x, \theta_1, \theta_2) = 0$. In this case we put $n = 1/2$. Under this assumption, there is no control in the first equation of system (1.6) and is weak in the other equations. An interesting example is considered in Section 2.

It follows from conditions (1.11) and (1.12) that

$$u^* = U_0 \operatorname{sign} G^*(\varphi, \theta) \operatorname{sign} \beta(\varphi) \tag{1.22}$$

as $\mu \to 0$. In the admissible domain Σ the variable φ changes between the phases φ^* and φ^s, corresponding to the minimum and maximum of the potential function $U(\varphi)$. In particular, this implies that

$$\operatorname{sign} U_{\varphi}(\varphi) = \operatorname{sign}(\varphi - \varphi^*) \tag{1.23}$$

in the admissible domain. Considering the equality $U_{\varphi}(\varphi) = -\beta(\varphi)$ and taking into account relations (1.7), (1.22) and (1.23), we find

$$\begin{aligned} u^* &= -U_0 \operatorname{sign} G^*(\varphi, \theta) \operatorname{sign}(\varphi - \varphi^*) \\ u^0(x, \theta_1, \theta_2) &= -U_0 \operatorname{sign}[m_1 G_1(x, \theta_1, \theta_2) + \\ &+ m_2 G_2(x, \theta_1, \theta_2)] \operatorname{sign}(\varphi - \varphi^*), \quad \varphi = m_1\theta_1 + m_2\theta_2 \end{aligned} \tag{1.24}$$

Substituting control (1.24) into Eq. (1.6) and averaging all terms except the perturbation, we obtain the partially averaged system

$$\begin{aligned} \dot{\upsilon}_0 &= \mu[\beta(\varphi_0) + \Delta^*(\varphi_0, \theta, \xi(t))] \\ \dot{\varphi}_0 &= \mu[\upsilon_0 - U_0 g_0(\varphi_0) \operatorname{sign}(\varphi_0 - \varphi^*)] \\ \dot{\theta} &= \omega^* + \mu\Omega\upsilon_0 \end{aligned} \tag{1.25}$$

where $g_0(\varphi) = \langle |G^*(\varphi, \theta)| \rangle$, that is $g_0(\varphi) \ge 0$. In general, it is difficult to estimate the limiting control performance. However, if condition (1.17) holds, the averaging procedure results in the reduced system

$$\dot{\upsilon}_0 = \mu\beta(\varphi_0), \quad \dot{\varphi}_0 = \mu[\upsilon_0 - U_0 g_0(\varphi_0) \operatorname{sign}(\varphi_0 - \varphi^*)] \tag{1.26}$$

Taking into account that $\beta(\varphi) = -U_{\varphi}(\varphi)$ and using relations (1.23), we write the equation for the change in total energy

$$\dot{E} = \mu[-\beta(\varphi_0)\dot{\varphi}_0 + \upsilon_0\dot{\upsilon}_0] = -\mu U_0 g_0(\varphi_0)|\beta(\varphi_0)| < 0 \tag{1.27}$$

It follows from inequality (1.27) that $\varphi \to \varphi^*$ and $\upsilon \to 0$ as $t \to \infty$ if condition (1.17) holds.

Under the assumptions made, the control directly affects the phase detuning φ, whereas the frequency detuning ν satisfies Eq. (1.25). We define a measure of the phase detuning as

$$H^{\varphi} = (\varphi - \varphi^*)^2/2 \tag{1.28}$$

We construct a control u^{φ}, which minimizes the derivative $\dot{H}^{\varphi}$ at each instant t under the constraint $|u| \le U_0$. Let $n = 1/2$. Differentiating the function (1.28) by virtue of Eq. (1.6) we obtain

$$\dot{H}^{\varphi} = \mu\dot{\varphi}(\varphi - \varphi^*) = \mu[v + G^*(\varphi, \theta)u](\varphi - \varphi^*) \tag{1.29}$$

as $\mu \to 0$. The higher-order terms are neglected. Minimizing the right-hand side of Eq. (1.29) under the constraint $|u| \le U_0$ we obtain

$$u^{\varphi} = u^* = -U_0 \operatorname{sign} G^*(\varphi, \theta) \operatorname{sign}(\varphi - \varphi^*) \tag{1.30}$$

Control (1.30) and correspondingly, controls (1.24) can be interpreted as phase controls.

Equalities (1.21) and (1.30) imply that the general criterion (1.10) can be replaced by criterion (1.19) or (1.28), depending on the system structure and the type of control.

2. FREQUENCY CONTROL OF THE FORCED MOTION OF COUPLED OSCILLATORY SYSTEMS

We will now suppose that a resonance circuit, loosely coupled to the input of the non-linear system, enhances a weak periodic excitation. The purpose of the control is to maintain the oscillation frequency in the non-linear system when there are perturbations. Perturbations can appear in the system directly, as well as in the resonance and control circuits. We will investigate the feasibility of different phase and frequency control schemes.

A control which directly affects the non-linear system. The equations of motion have the form

$$\begin{aligned} &\ddot{\psi} + \varepsilon b\dot{\psi} + \Omega^2\psi + \varepsilon g_1(\psi, \xi_1(t)) = \varepsilon a \sin\Omega t + \varepsilon s(x, \dot{x}) \\ &\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon g_2(x, \xi_2(t)) = \varepsilon q(\psi, \dot{\psi}) + \varepsilon u \end{aligned} \tag{2.1}$$

Here $\phi(x) = d\Pi(x)/dx$, where $\Pi(x)$ is the potential function of the conservative part of the system. The perturbations $\xi_{1,2}(t)$ satisfy the assumptions of Section 1. The terms $q(\psi, \dot{\psi})$ and $s(x, \dot{x})$ describe the interaction of the subsystems. The control u is designed following the criteria of Section 1.

We reduce system (2.1) to standard form. We put $\dot{x} = z$ and introduce the new variables y and θ_2 by the formulae [1]

$$y = \frac{1}{2}z^2 + \Pi(x), \quad \frac{\partial\theta_2}{\partial x} = \frac{\omega(y)}{z(y, x)}, \quad z(y, x) = \pm\sqrt{2(y - \Pi(x))}, \quad \omega(y) = \frac{2\pi}{T(y)} \tag{2.2}$$

where

$$T(y) = \oint_{y = \text{const}} \frac{dx}{z(y, x)}$$

Relations (2.2) define the functions $x = X(y, \theta_2)$ and $\dot{x} = z(y, x) = Z(y, \theta_2)$. We use the standard replacement of the variables ψ and $\dot{\psi}$

$$\psi = R\cos\theta_1, \quad \dot{\psi} = -\Omega R \sin\theta_1 \tag{2.3}$$

Substituting the new variables (2.2) and (2.3) into Eq. (2.1) and using the notation of Section 1, we reduce system (2.1) to the form

$$\begin{aligned} \dot{R} &= -\frac{\varepsilon}{\Omega}[\Psi(R, \theta_1, \theta_3) + S(y, \theta_2) + \Delta_1(R, \theta_1, \xi_1(t))]\sin\theta_1 \\ \dot{y} &= \varepsilon\{f(y, \theta_2) + [Q(R, \theta_1) + u]Z(y, \theta_2) + \Delta_2(y, \theta_2, \xi_2(t))\} \\ \dot{\theta}_1 &= \Omega - \frac{\varepsilon}{\Omega R}[\Psi(R, \theta_1, \theta_3) + S(y, \theta_2) + \Delta_1(R, \theta_1, \xi(t))]\cos\theta_1 \\ \dot{\theta}_2 &= \omega(y) + \varepsilon\frac{\partial\omega}{\partial y}\{f(y, \theta_2) + [Q(R, \theta_1) + u]Z(y, \theta_2) + \Delta_2(y, \theta_2, \xi_2(t))\} \\ \dot{\theta}_3 &= \Omega \end{aligned} \tag{2.4}$$

where

$$\Psi(R,\theta_1,\theta_3) = b\Omega R\sin\theta_1 + a\sin\theta_3, \quad f(y,\theta_2) = -nZ^2(y,\theta_2)$$
$$S(y,\theta_2) = s(X(y,\theta_2), Z(y,\theta_2)), \quad Q(R,\theta_1) = q(R\cos\theta_1, -\Omega R\sin\theta_1)$$
$$\Delta_1(R,\theta_1,\xi_1(t)) = -g_1(R\cos\theta_1,\xi_2(t)) \tag{2.5}$$
$$\Delta_2(y,\theta_2,\xi_2(t)) = -g_2(X(y,\theta_2),\xi_2(t))Z(y,\theta_2)$$

The non-linear terms in the equations of motion generate an infinite number of resonance relations of the type $n\omega(y) = m\Omega$. Suppose the objective is to maintain the resonance oscillations corresponding to the first harmonic. In this case conditions (1.2) and (1.3) take the form

$$\rho(y^*) = \omega(y^*) - \Omega = 0, \quad d\rho(y^*)/dy = d\omega(y^*)/dy = r \neq 0 \tag{2.6}$$

We will use transformation (1.4) to analyse the motion in the neighbourhood of resonance. We introduce the new variables

$$\varphi = \theta_2 - \theta_3, \quad \varphi_1 = \theta_1 - \theta_3, \quad \theta_3 = \theta$$
$$\mu\upsilon = \rho(y) = \omega(y) - \Omega, \quad \mu = \varepsilon^{1/2} \tag{2.7}$$

Substituting relations (2.6) and (2.7) into Eq. (2.4) and neglecting the unimportant higher-order terms, we obtain the equations

$$\dot{R} = \mu^2 P_1(R,\varphi,\varphi_1,\theta,\xi_1(t)), \quad \dot{\varphi}_1 = \mu^2 P_2(R,\varphi,\varphi_1,\theta,\xi_1(t))$$
$$\dot{\upsilon} = \mu[f^*(\varphi,\theta) + F^*(\varphi,\theta)u + Y^*(R,\varphi,\theta) + \Delta_2^*(\theta,\xi_2(t))] =$$
$$= \mu[F^*(\varphi,\theta)u + V(R,\varphi,\theta,\xi_2(t))] \tag{2.8}$$
$$\dot{\varphi} = \mu\upsilon + \mu^2 rV(R,\varphi,\theta,\xi_2(t)), \quad \dot{\theta} = \Omega$$

where

$$Y^* = r^{-1}Q(R,\theta+\varphi_1)Z(y^*,\theta+\varphi), \quad F^* = r^{-1}Z(y^*,\theta+\varphi)$$

The coefficients f^*, F^* and Δ_2^* are defined in the same way as in Section 1. The coefficients $P_{1,2}$ are obtained by substituting $y = y^*$, $\theta_1 = \theta + \varphi_1$, $\theta_2 = \theta + \varphi$ into the right-hand sides of the corresponding equations of system (2.4). The precise form of the terms $P_{1,2}$ is unimportant for the further transformations.

Since the coefficient V in the third equation depends on the slow variable R, system (2.8) does not allow of a partition of the conservative subsystem similar to (1.8). However, Eq. (2.8) implies that the control u directly affects the frequency detuning ν. Hence the control can be based on criterion (1.20). By the same arguments as above, we obtain the frequency control

$$u^\upsilon = -U_0 \operatorname{sign} F^*(\varphi,\theta) \operatorname{sign} \upsilon \tag{2.9}$$

which is identical with (1.21). Since

$$F^*(\varphi,\theta) = r^{-1}Z(y^*,\theta+\varphi), \quad Z(y,\theta+\varphi) = \dot{x}$$

the feedback control takes the form

$$u^\upsilon = -U_0 \operatorname{sign}(r^{-1}\dot{x}) \operatorname{sign}[\omega(y) - \Omega] \tag{2.10}$$

This solution means that, under the assumptions made, the near-optimal control counteracts the frequency detuning and is independent of the perturbation and the parameters of the linear subsystem. The only parameter necessary for constructing the control is sign = sign $\omega_y(y^*)$. This parameter can be found without calculating the frequency $\omega(y)$, namely, $r > 0$ if the system is "hard", and $r < 0$ if the system is "soft" in the neighbourhood of the point y^*. Note that the control depends on both the slow frequency detuning ν and on the fast variable $\dot{x}$.

Control of the excitation frequency. Suppose a control cannot be applied to the system as an external force or feedback but can change the excitation frequency. In this case the equations of the controlled motion take the form

$$\begin{aligned}
&\ddot{\psi} + \varepsilon b\dot{\psi} + \Omega^2\psi + \varepsilon g_1(x, \xi_1(t)) = \varepsilon a \sin\theta_3 + \varepsilon s(x, \dot{x}) \\
&\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon g_2(x, \xi_2(t)) = \varepsilon q(\psi, \dot{\psi}) \\
&\dot{\theta}_3 = \Omega + \varepsilon^{1/2} u
\end{aligned} \tag{2.11}$$

On the right-hand sides of Eqs (2.11) we have used the same notation as in Eq. (2.1). The changes of variables (2.2) and (2.3) transform Eqs (2.11) to the standard form, which is identical with Eqs (2.4), if we replace the last equation of system (2.4) by the corresponding equation of system (2.11) and put $u = 0$ in the other equations.

Formula (2.6) and (2.7) define the resonance relation and the change of variables in the near-resonance region. The equations of motion in the new variables take the form

$$\begin{aligned}
&\dot{R} = \mu^2 P_1, \quad \dot{\varphi}_1 = -\mu u + \mu^2 P_2 \\
&\dot{\upsilon} = \mu V, \quad \dot{\varphi} = \mu \upsilon - \mu u + \mu^2 r V, \quad \dot{\theta} = \Omega + \mu u
\end{aligned} \tag{2.12}$$

The functions $P_{1,2}$ and V are defined as in system (2.8).

The control u occurs in three equations of system (2.12). If the problem is to maintain the resonance oscillations of the non-linear subsystem regardless of the resonance circuit dynamics, we can construct a control which minimizes criterion (1.28). Arguing as above, using formula (1.30), and considering $G^* = -1$ in system (2.12), we obtain

$$u^{\varphi} = U_0 \operatorname{sign}(\varphi - \varphi^*) \tag{2.13}$$

Control (2.13) counteracts the phase detuning and is independent of the perturbation and the parameters of the non-linear system and the resonance circuit.

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